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A SIMPLE EXTENSION OF STOLLMANN'S LEMMA
TO CORRELATED POTENTIALS

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Abstract

We propose a fairly simple and natural extension of Stollmann's lemma to correlated random variables, described earlier in [8]. This extension allows (just as the original Stollmann's lemma does) to obtain Wegner-type estimates even in some problems of spectral analysis of random operators where the Wegner's lemma is inapplicable (e.g. for multi-particle Hamiltonians).

To the best of author's knowledge, such an extension seems to be original. However, the author will appreciate any reference to articles or preprints where similar results are proved.

1 Introduction

The regularity problem for the limiting distribution of eigen-values of infinite dimensional self-adjoint operators appears in many problems of mathematical physics. Specifically, consider a lattice Schrödinger operator (LSO, for short) $H : \ell^2(\mathbb{Z}^d) \rightarrow \ell^2(\mathbb{Z}^d)$ given by

$$(H\psi)(x) = \sum_{y: |y-x|=1} \psi(y) + V(x)\psi(x); \quad x, y \in \mathbb{Z}^d.$$

For each finite subset $\Lambda \subset \mathbb{Z}^d$, let $E_j^\Lambda, j = 1, \dots, |\Lambda|$, be eigen-values of H with Dirichlet b.c. in Λ . Consider the family of finite sets $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ and define the following quantity (which does not necessarily exist for an arbitrary LSO):

$$k(E) = \lim_{L \rightarrow \infty} \frac{1}{(2L+1)^d} \text{card} \left\{ j : E_j^{\Lambda_L} \leq E \right\}.$$

If the above limit exists, $k(E)$ is called the limiting distribution function (LDF) of e.v. of H . One can easily construct various examples of the function $V : \mathbb{Z}^d \rightarrow \mathbb{R}$ (called potential of the operator H) for which the LDF does not exist. One can prove the existence of LDF for periodic potentials V , but even in this, relatively simple situation existence of $k(E)$ is not a trivial fact.

However, one can prove existence of $k(E)$ in a large class of *ergodic random* potentials. Namely, consider an ergodic dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{T^x, x \in \mathbb{Z}^d\})$ with discrete time \mathbb{Z}^d and a measurable function (sometimes called a hull) $v : \Omega \rightarrow \mathbb{R}$. Then we can introduce a family of sample potentials

$$V(x, \omega) = v(T^x \omega), \quad x \in \mathbb{Z}^d,$$

labeled by $\omega \in \Omega$. Under the assumption of ergodicity of $\{T^x\}$, the quantity

$$k(E, \omega) = \lim_{L \rightarrow \infty} \frac{1}{(2L+1)^d} \text{card} \left\{ j : E_j^{\Lambda_L}(\omega) \leq E \right\}$$

is well-defined \mathbb{P} -a.s. Moreover, $k(E, \omega)$ is \mathbb{P} -a.s. independent of ω , so its value taken for a.e. ω is natural to take as $k(E)$. In such a context, $k(E)$ is usually called *integrated density of states* (IDS, for short). It admits an equivalent definition:

$$k(E) = \mathbb{E} \left[(f, \Pi_{(-\infty, E]}(H(\omega)f)) \right],$$

where $f \in \ell^2(\mathbb{Z}^d)$ is any vector of unit norm, and $\Pi_{(-\infty, E]}(H(\omega))$ is the spectral projection of $H(\omega)$ on $(-\infty, E]$. The reader can find a detailed discussion of the existence problem of IDS in excellent monographs by Carmona and Lacroix [4] and by Pastur and Figotin [16].

It is not difficult to see that $k(E)$ can be considered as the distribution function of a normalized measure, i.e. probability measure, on \mathbb{R} . If this measure $dK(E)$, called measure of states, is absolutely continuous with respect to Lebesgue measure dE , its density (or Radon–Nikodim derivative) $dK(E)/dE$ is called the density of states (DoS). In physical literature, it is customary to neglect the problem of existence of such density, for if $dK(E)/dE$ is not a function, then "it is simply a generalized function". However, the real problem is not terminological. The actual, explicit estimates of the probabilities of the form

$$\mathbb{P} \left\{ \exists \text{ eigen-value } E_j^{\Lambda_L} \in (a, a + \epsilon) \right\}$$

for LSO H_{Λ_L} in a finite cube Λ_L of size L , for small ϵ , often depend essentially upon the existence and the regularity properties of the DoS $dk(E)/dE$.

Apparently, the first fairly general result relative to existence and boundedness of the DoS is due to Wegner [18]. Traditionally referred to as Wegner's *lemma*, it certainly deserves to be called *theorem*.

Theorem 1 (Wegner) *Assume that $\{V(x, \omega), x \in \mathbb{Z}^d\}$ are i.i.d. r.v. with bounded density $p_V(u)$ of their common probability distribution: $\|p_V\|_\infty = C < \infty$. Then the DoS $dk(E)/dE$ exists and is bounded by the same constant C .*

The proof can be found, for example, in the monograph [4].

This estimate and some of its generalizations have been used in the multi-scale analysis (MSA) developed in the works by Fröhlich and Spencer [13], Fröhlich, Spencer, Martinelli and Scoppola [14], von Dreifus and Klein [11], [12], Aizenman and Molchanov [1], and in a number of more recent works where the so-called Anderson Localization phenomenon has been observed. Namely, it has been proven that all e.f. of random lattice Schrödinger operators decay exponentially at infinity with probability one (for \mathbb{P} -a.e. sample of random potential $V(\omega)$). Von Dreifus and Klein [12] proved an analog of Wegner estimate and used it in their proof of localization for Gaussian and some other correlated (but non-deterministic) potentials. The author of these lines recently proved, in a joint work with Yu. Suhov [9], an analog of Wegner estimate for a system of two or more interacting quantum particles on the lattice under the assumption of analyticity of the probability density $p_V(u)$, using a rigorous path integral formula by Molchanov (see a detailed discussion of this formula in the monograph [4]). In order to relax the analyticity assumption in a multi-particle context, V.C. and Yu. Suhov later used ([10]) a more general and flexible result guaranteeing existence and boundedness of the DoS: the Stollmann's lemma, which we discuss below.

In the present work, we propose a fairly simple and natural extension of Stollmann's lemma to correlated, but still non-deterministic random fields generating random potentials. *To the best of author's knowledge, such an extension seems to be original, although very simple. However, the author will appreciate any reference to published papers or preprints where the same or similar result was mentioned and proved.* Our main motivation here is to lay out a way to interesting applications to localization problems for multi-particle systems.

2 Stollmann's lemma for product measures

Recall the Stollmann's lemma and its proof for independent r.v. Let $m \geq 1$ be a positive integer, and J an abstract finite set with $|J| (= \text{card} J) = m$. Consider the Euclidean space $\mathbb{R}^J \cong \mathbb{R}^m$ with standard basis (e_1, \dots, e_m) , and its positive quadrant

$$\mathbb{R}_+^J = \left\{ q \in \mathbb{R}^J : q_j \geq 0, j = 1, 2, \dots, m \right\}.$$

For any measure μ on \mathbb{R} , we will denote by μ^m the product measure $\mu \times \dots \times \mu$ on \mathbb{R}^J . Furthermore, for any probability measure μ and for any $\epsilon > 0$, define the following quantity:

$$s(\mu, \epsilon) = \sup_{a \in \mathbb{R}} \int_a^{a+\epsilon} d\mu(t)$$

and assume that $s(\mu, \epsilon)$ is finite. Furthermore, let μ^{m-1} be the marginal probability distribution induced by μ^m on $q' = (q_2, \dots, q_m)$.

Definition 1 Let J be a finite set with $|J| = m$. Consider a function $\Phi : \mathbb{R}^J \rightarrow \mathbb{R}$ on \mathbb{R}^J which we will identify with \mathbb{R}^m . Function Φ is called J -monotonic if it satisfies the following conditions:

(1) for any $r \in \mathbb{R}_+^m$ and any $q \in \mathbb{R}^m$,

$$\Phi(q + r) \geq \Phi(q); \tag{1}$$

(2) moreover, for $e = e_1 + \dots + e_m \in \mathbb{R}^m$, for any $q \in \mathbb{R}^m$ and for any $t > 0$

$$\Phi(q + t \cdot e) - \Phi(q) \geq t. \tag{2}$$

It is convenient to introduce the notion of J -monotonic operators considered as quadratic forms. In the following definition, we use the same notations as above.

Definition 2 Let \mathcal{H} be a Hilbert space. A family of self-adjoint operators $B : \mathcal{H} \times \mathbb{R}^J \rightarrow \mathcal{H}$ is called J -monotonic if, for any vector $f \in \mathcal{H}$ with $\|f\| = 1$, the function $\Phi_f : \mathbb{R}^J \rightarrow \mathbb{R}$ defined by

$$\Phi_f(q) = (B(q)f, f)$$

is monotonic.

In other words, the quadratic form $Q_{B(q)}(f) := (B(q)f, f)$ as function of $q \in \mathbb{R}^J$ is non-decreasing in any q_j , $j = 1, \dots, |J|$, and

$$(B(q + t \cdot e)f, f) - (B(q)f, f) \geq t \cdot \|f\|^2.$$

Remark 1 By virtue of the variational principle for self-adjoint operators, if an operator family $H(q)$ in a finite-dimensional Hilbert space \mathcal{H} is J -monotonic, then each eigen-value $E_k^{B(q)}$ of $B(q)$ is a J -monotonic function.

Remark 2 If $H(q)$, $q \in \mathbb{R}^J$, is a J -monotonic operator family in Hilbert space \mathcal{H} , and $H_0 : \mathcal{H} \rightarrow \mathcal{H}$ is an arbitrary self-adjoint operator, then the family $H_0 + H(q)$ is also J -monotonic.

This explains why the notion of monotonicity is relevant to spectral theory of random operators. Note also, that this property can be easily extended to physically interesting examples where \mathcal{H} has infinite dimension, but $H(q)$ have, e.g., compact resolvent, as in the case of Schrödinger operators in a finite cube with Dirichlet b.c. and with bounded potential, so the respective spectrum is pure point, and even discrete.

Theorem 2 (Stollmann, [17]) *Let J be a finite index set, $|J| = m$, μ be a probability measure on \mathbb{R} , and μ^m be the product measure on \mathbb{R}^J with marginal measures μ . If the function $\Phi : \mathbb{R}^J \rightarrow \mathbb{R}$ is J -monotonic, then for any open interval $I \subset \mathbb{R}$ we have*

$$\mu^m \{ q : \Phi(q) \in I \} \leq m \cdot s(\mu, |I|).$$

Proof. Let $I = (a, b)$, $b - a = \epsilon > 0$, and consider the set

$$A = \{ q : \Phi(q) \leq a \}.$$

Furthermore, define recursively sets A_j^ϵ , $j = 0, \dots, m$, by setting

$$A_0^\epsilon = A, \quad A_j^\epsilon = A_{j-1}^\epsilon + [0, \epsilon] e_j := \{ q + t e_j : q \in A_{j-1}^\epsilon, t \in [0, \epsilon] \}.$$

Obviously, the sequence of sets A_j^ϵ , $j = 1, 2, \dots$, is increasing with j . The monotonicity property implies

$$\{ q : \Phi(q) < b \} \subset A_m^\epsilon.$$

Indeed, if $\Phi(q) < b$, then for the vector $q' := q - \epsilon \cdot e$ we have by (2):

$$\Phi(q') \leq \Phi(q' + \epsilon \cdot e) - \epsilon = \Phi(q) - \epsilon \leq b - \epsilon \leq a,$$

meaning that $q' \in \{ \Phi \leq a \} = A$ and, therefore,

$$q = q' + \epsilon \cdot e \in A_m^\epsilon.$$

Now, we conclude that

$$\begin{aligned} \{ q : \Phi(q) \in I \} &= \{ q : \Phi(q) \in (a, b) \} \\ &= \{ q : \Phi(q) < b \} \setminus \{ q : \Phi(q) \leq a \} \subset A_m^\epsilon \setminus A. \end{aligned}$$

Furthermore,

$$\begin{aligned} \mu^m \{ q : \Phi(q) \in I \} &\leq \mu^m (A_m^\epsilon \setminus A) \\ &= \mu^m \left(\bigcup_{j=1}^m (A_j^\epsilon \setminus A_{j-1}^\epsilon) \right) \leq \sum_{j=1}^m \mu^m (A_j^\epsilon \setminus A_{j-1}^\epsilon). \end{aligned}$$

For $q' \in \mathbb{R}^{m-1}$, set

$$I_1(q') = \{ q_1 \in \mathbb{R} : (q_1, q') \in A_1^\epsilon \setminus A \}.$$

By definition of set A_1^ϵ , this is an interval of length not bigger than ϵ . Then we have

$$\mu^m (A_1^\epsilon \setminus A) = \int d\mu^{m-1}(q') \int_{I_1} d\mu(q_1) \leq s(\mu, \epsilon). \quad (3)$$

Similarly, we obtain for $j = 2, \dots, m$

$$\mu^m (A_j^\epsilon \setminus A_{j-1}^\epsilon) \leq s(\mu, \epsilon),$$

yielding

$$\mu^m \{ q : \Phi(q) \in I \} \leq \sum_{j=1}^m \mu^m (A_j^\epsilon \setminus A_{j-1}^\epsilon) \leq m \cdot s(\mu, \epsilon). \quad \blacksquare$$

Now, taking into account the above Remark 1, Stollmann's theorem yields immediately the following estimate.

Theorem 3 Let H_Λ be an LSO with random potential $V(x; \omega)$ in a finite box $\Lambda \subset \mathbb{Z}^d$ with Dirichlet b.c., and $\Sigma(H_\Lambda)$ its spectrum, i.e. the collection of its eigen-values $E_j^{(\Lambda)}$, $j = 1, \dots, |\Lambda|$. Assume that r.v. $V(x; \cdot)$ are i.i.d. with marginal distribution function F_V satisfying

$$s(\epsilon) = \sup_{a \in \mathbb{R}} (F_V(a + \epsilon) - F_V(a)) < \infty.$$

Then

$$\mathbb{P} \{ \text{dist}(\Sigma(H_\Lambda(\omega)), E) \leq \epsilon \} \leq |\Lambda|^2 s(\epsilon).$$

3 Extension to multi-particle systems

Results of this section have been obtained by the author and Y. Suhov [9].

Let $N > 1$ and $d \geq 1$ be two positive integers and consider a random LSO $H = H(\omega)$ which can be used, in the framework of tight-binding approximation, as the Hamiltonian of a system of N quantum particles in \mathbb{Z}^d with random external potential V and interaction potential U . Specifically, let $x_1, \dots, x_N \in \mathbb{Z}^d$ be positions of quantum particles in the lattice \mathbb{Z}^d , and $\underline{x} = (x_1, \dots, x_N)$. Let $\{V(x; \omega), x \in \mathbb{Z}^d\}$ be a random field on \mathbb{Z}^d describing the external potential acting on all particles, and $U : (x_1, \dots, x_N) \mapsto \mathbb{R}$ be the interaction energy of the particles. In physics, U is usually to be symmetric function of its N arguments $x_1, \dots, x_N \in \mathbb{Z}^d$. We will assume in this section that the system in question obeys either Fermi or Bose quantum statistics, so it is convenient to assume U to be symmetric. Note, however, that the results of this section can be extended, with natural modifications, to more general interactions U . Further, in [9] U is assumed to be finite-range interaction:

$$\text{supp } U \subset \{ \underline{x} : \max(|x_j - x_k| \leq r) \}, \quad r < \infty.$$

Such an assumption is required in the proof of Anderson localization for multi-particle systems, however, it is irrelevant to the Wegner–Stollmann estimate we are going to discuss below.

Now, let H be as follows:

$$(H(\omega)f)(\underline{x}) = \sum_{j=1}^N \left(\Delta^{(j)} + V(x_j; \omega) \right) f(\underline{x}) + U(\underline{x}),$$

where $\Delta^{(j)}$ is the lattice Laplacian acting on the j -th particle, i.e.

$$\Delta^{(j)} = \mathbf{1}_1 \otimes \dots \otimes \Delta_j \otimes \dots \otimes \mathbf{1}_N$$

acting in Hilbert space $\ell^2(\mathbb{Z}^{Nd})$. For any finite "box"

$$\Lambda = \Lambda^{(1)} \times \dots \times \Lambda^{(N)} \subset \mathbb{Z}^{Nd}$$

one can consider the restriction, $H_\Lambda(\omega)$, of $H(\omega)$ on Λ with Dirichlet b.c. It is easy to see that the potential

$$W(\underline{x}) = \sum_{j=1}^N V(x_j; \omega) + U(\underline{x})$$

is no longer an i.i.d. random field on \mathbb{Z}^{Nd} , even if V is i.i.d. Therefore, neither Wegner's nor Stollmann's *estimate* does not apply *directly*. But, in fact, Stollmann's lemma *does* apply to multi-particle systems, virtually in the same way as to single-particle ones.

Theorem 4 Assume that r.v. $V(x; \cdot)$ are i.i.d. with marginal distribution function F_V satisfying

$$s(\epsilon) = \sup_{a \in \mathbb{R}} (F_V(a + \epsilon) - F_V(a)) < \infty.$$

Then

$$\mathbb{P} \{ \text{dist}(\Sigma(H_\Lambda(\omega), E) \leq \epsilon \} \leq |\Lambda| \cdot M(\Lambda) \cdot s(\epsilon),$$

with

$$M(\Lambda) = \sum_{j=1}^N \text{card } \Lambda^{(j)}.$$

Proof. Fix Λ and consider the union of all lattice points in \mathbb{Z}^d which belong to the single-particle projections $\Lambda^{(j)}$, $j = 1, \dots, N$:

$$\mathcal{X}(\Lambda) = \bigcup_{j=1}^N \Lambda^{(j)} \subset \mathbb{Z}^d.$$

Now we can apply Stollmann's lemma to H_Λ by taking the index set $J = \mathcal{X}(\Lambda)$ and auxiliary probability space \mathbb{R}^J . Indeed, the random potential $\hat{V}(\underline{x}; \omega) := V(x_1; \omega) + \dots + V(x_N; \omega)$ can be re-written as follows:

$$V(x_1; \omega) + \dots + V(x_N; \omega) = \sum_{y \in \mathcal{X}(\Lambda)} c(\underline{x}, y) V(y; \omega)$$

with integer coefficients $c(\underline{x}, y)$ such that

$$c(\underline{x}, y) \geq 0, \quad \sum_{y \in \mathcal{X}(\Lambda)} c(\underline{x}, y) = N. \quad (4)$$

For example, if $N = 2$, one can have either $V(x_1; \omega) + V(x_2; \omega)$ with $x_1 \neq x_2$, in which case we have

$$c(\underline{x}, y) = \begin{cases} 1, & \text{if } y = x_1 \text{ or } y = x_2 \\ 0, & \text{otherwise} \end{cases}$$

or $V(x_1; \omega) + V(x_1; \omega) = 2V(x_1; \omega)$ for "diagonal" points (x_1, x_2) , where

$$c(\underline{x}, y) = \begin{cases} 2, & \text{if } y = x_1 \\ 0, & \text{otherwise} \end{cases}$$

In any case, as shows (4), random potential at $\underline{x} \in \Lambda$ is a linear function of one or more coordinates in the auxiliary space \mathbb{R}^J growing at rate $\geq Nt \geq t$ along the principal diagonal $\{q_1 = q_2 = \dots = q_{|J|} = t \in \mathbb{R}\}$. Hence, the operators of multiplication by $\hat{V}(\underline{x}; \omega)$ form a J -monotonic family, and, by virtue of Remark 2, the same holds for $H = H_0 + U + \hat{V}(\omega)$, just as in the single-particle case (and even "better", for $N > 1$!). By Theorem 2, this implies immediately the estimate

$$\mathbb{P} \{ \text{dist}(\Sigma(H_\Lambda(\omega), E) \leq \epsilon \} \leq |\Lambda|^2 s(\mu, \epsilon). \quad \blacksquare$$

It is not difficult to see that the same argument, with obvious notational modifications, applies to Fermi and Bose lattice quantum systems, i.e. to restrictions of H to the subspaces of symmetric (Bose case) or anti-symmetric (Fermi case) functions of N arguments x_1, \dots, x_N on $(\mathbb{Z}^d)^N$.

4 Extension to correlated random variables

Now let μ^m be a measure on \mathbb{R}^m with marginal distributions of order $m - 1$,

$$\mu_j^{m-1}(q'_{\neq j}) = \mu_j^{m-1}(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_m), \quad j = 1, \dots, m,$$

and conditional distributions $\mu_j^1(q_j | q'_{\neq j})$ on q_j given all $q_k, k \neq j$. For every $\epsilon > 0$, define the following quantity:

$$C_1(\mu^m, \epsilon) = \max_j \sup_{a \in \mathbb{R}} \int d\mu^{m-1}(q'_{\neq j}) \int_a^{a+\epsilon} d\mu(q_1 | q'_{\neq j})$$

and assume that $C_1(\mu, \epsilon)$ is finite:

$$\max_j \sup_{a \in \mathbb{R}} \int d\mu^{m-1}(q'_{\neq j}) \int_a^{a+\epsilon} d\mu(q_1 | q'_{\neq j}) < \infty. \quad (5)$$

Remark 3 As a simple sufficient condition of finiteness of $C_1(\mu, \epsilon)$, one can use, e.g., a uniform continuity (but not necessarily **absolute continuity** !) of the single-point conditional distributions,

$$\max_j \sup_{q'_{\neq j}} \sup_{a \in \mathbb{R}} \int_a^{a+\epsilon} d\mu(q_j | q'_{\neq j}) \leq C_2(\mu^m, \epsilon) < \infty$$

or even the existence and uniform boundedness of the **density** $p(q_j | q'_{\neq j})$ of these conditional distributions:

$$\sup_{q_j \in \mathbb{R}} p(q_j | q'_{\neq j}) \leq C_3(\mu^m, \epsilon).$$

Remark 4 In applications to localization problems, the aforementioned continuity moduli $C_1(\mu^m, \epsilon)$, $C_2(\mu^m, \epsilon)$, $C_3(\mu^m, \epsilon)$ need to decay not too slowly as $\epsilon \rightarrow 0$. A power decay of order $O(\epsilon^\beta)$ with $\beta > 0$ is certainly sufficient, but it can be essentially relaxed. For example, it suffices to have an upper bound of the form

$$C_1(\mu^m, e^{-L^\beta}) \leq \text{Const} \cdot L^{-B},$$

uniformly for all sufficiently large $L > 0$ with some (arbitrarily small) $\beta > 0$ and with $B > 0$ which should be sufficiently big, depending on the specific spectral problem.

Using notations of the previous section, one can formulate the following generalization of Stollmann's lemma.

Lemma 1 Let $\Phi : \mathbb{R}^J \rightarrow \mathbb{R}$, $\mathbb{R}^J \cong \mathbb{R}^m$, be a J -monotonic function and μ^m a probability measure on $\mathbb{R}^m \cong \mathbb{R}^J$ with $C_1(\mu^m, \epsilon) < \infty$. Then for any interval $I \subset \mathbb{R}$ of length $|I| = \epsilon > 0$, we have

$$\mu^m \{ q : \Phi(q) \in I \} \leq m \cdot C_1(\mu, \epsilon).$$

Proof. We proceed as in the proof of Stollmann's lemma and introduce in \mathbb{R}^m the sets $A = \{ q : \Phi(q) \leq a \}$ and A_j^ϵ , $j = 0, \dots, m$. Here, again, we have

$$\{ q : \Phi(q) \in I \} \subset A_m^\epsilon \setminus A$$

and

$$\mu^m \{ q : \Phi(q) \in I \} \leq \sum_{j=1}^m \mu^m (A_j^\epsilon \setminus A_{j-1}^\epsilon).$$

For $q'_{\neq 1} \in \mathbb{R}^{m-1}$, we set

$$I_1(q'_{\neq 1}) = \{q_1 \in \mathbb{R} : (q_1, q'_{\neq 1}) \in A_1^\epsilon \setminus A\}.$$

Furthermore, we come to the following upper bound which generalizes (3):

$$\mu^m(A_1^\epsilon \setminus A) = \int d\mu^{m-1}(q') \int_{I_1} d\mu(q_1|q') \leq C_1(\mu, \epsilon). \quad (6)$$

Similarly, we obtain for $j = 2, \dots, m$

$$\mu^m(A_j^\epsilon \setminus A_{j-1}^\epsilon) \leq C_1(\mu, \epsilon),$$

yielding

$$\mu^m\{q : \Phi(q) \in I\} \leq \sum_{j=1}^m \mu^m(A_j^\epsilon \setminus A_{j-1}^\epsilon) \leq m \cdot C_1(\mu, \epsilon). \quad \blacksquare$$

5 Application to Gaussian random fields

Let $V(x, \omega), x \in \mathbb{Z}^d, d \geq 1$, be a regular stationary Gaussian field of zero mean on the lattice \mathbb{Z}^d . The regularity implies that the field $V(\cdot, \omega)$ is non-deterministic, i.e. the conditional probability distribution of $V(0, \cdot)$ given $\{V(y), y \neq 0\}$ is Gaussian with strictly positive variance. In other terms, the r.v. $V(0, \cdot)$, considered as a vector in the Hilbert space $\mathcal{H}_{V, \mathbb{Z}^d}$ generated by linear combinations of all $V(x, \cdot), x \in \mathbb{Z}^d$, with the scalar product

$$(\xi, \eta) = \mathbb{E}[\xi \eta],$$

does not belong to the subspace $\mathcal{H}_{V, \mathbb{Z}^d \setminus \{0\}}$:

$$\|V(0, \cdot) - \Pi_{\mathcal{H}_{V, \mathbb{Z}^d \setminus \{0\}}} V(0, \cdot)\|^2 = \tilde{\sigma}_0^2 > 0,$$

where

$$\Pi_{\mathcal{H}_{V, \mathbb{Z}^d}} \xi = \mathbb{E} \left[\xi \mid V(x, \cdot), x \in \mathbb{Z}^d \setminus \{0\} \right].$$

Furthermore, for any subset $\Lambda \subseteq \mathbb{Z}^d \setminus \{0\}$,

$$\|V(0, \cdot) - \Pi_{\mathcal{H}_{V, \Lambda}} V(0, \cdot)\|^2 \geq \tilde{\sigma}_0^2,$$

since

$$\mathcal{H}_{V, \Lambda} \subset \mathcal{H}_{V, \mathbb{Z}^d \setminus \{0\}}.$$

Therefore, the conditional variance of $V(0, \cdot)$ given any non-zero number of values of V outside $x = 0$ is bounded from below by $\tilde{\sigma}_0^2$. Respectively, the conditional probability density of $V(0, \cdot)$, for any such nontrivial condition is uniformly bounded by $(2\pi\tilde{\sigma}_0^2)^{-1/2} < \infty$. Now a direct application of Lemma 1 leads to the following statement.

Theorem 5 *Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset of the lattice, and $\Lambda' \subset \mathbb{Z}^d \setminus \Lambda$ any subset disjoint with Λ (Λ' may be empty). Consider a family of LSO $H_\Lambda(\omega)$ with Gaussian random potential $V(\omega)$ in Λ , with Dirichlet b.c. on $\partial\Lambda$. Then for any interval $I \subset \mathbb{R}$ of length $\epsilon > 0$, we have*

$$\mathbb{P} \{ \Sigma(H_\Lambda) \cap I \neq \emptyset \mid V(y, \cdot), y \in \Lambda' \} \leq C(V) |\Lambda|^2 \epsilon,$$

where the constant $C(V) < \infty$ whenever the Gaussian field V is non-deterministic.

6 Application to Gibbs fields with continuous spin

Apart from Gaussian fields, there exist several classes of random lattice fields for which the hypothesis of Lemma 1 can be easily verified. For example, conditional distributions of Gibbs fields are given explicitly in terms of their respective interaction potentials. Specifically, consider a lattice Gibbs field $s(x, \omega)$ with bounded continuous spin,

$$s : \Omega \times \mathbb{Z}^d \rightarrow \mathcal{S} = [a, b] \subset \mathbb{R}$$

generated by a short-range, bounded, two-body interaction potential $u(\cdot, \cdot)$. The spin space is assumed to be equipped with the Lebesgue measure ds . In other words, consider the formal Hamiltonian

$$H(s) = \sum_{x \in \mathbb{Z}^d} h(x) + \sum_{x \in \mathbb{Z}^d} \sum_{|y-x| \leq R} u_{|x-y|}(s(x), u(y)),$$

where $h : \mathcal{S} \rightarrow \mathbb{R}$ is the self-energy of a given spin. The interaction potentials $u_{|x-y|}(s(x), s(y))$ vanish for $|x - y| > R$ and are uniformly bounded:

$$\max_{l \leq R} \sup_{s, t \in \mathcal{S}} |u_l(s, t)| < \infty.$$

Then for any lattice point x and any configuration $s' = s'_{\neq x}$ of spins outside $\{x\}$, the *single-site* conditional distribution of $s(x)$ given the external configuration s' admits a *bounded density*

$$p(s_x | s'_{\neq x}) = \frac{e^{-\beta U(s_x | s')}}{\Xi(\beta, s')} = \frac{e^{-\beta U(s_x | s')}}{\int_{\mathcal{S}} e^{-\beta U(t | s')} dt}$$

with

$$U(s_x | s') := \sum_{y: |y-x| \leq R} u_{|x-y|}(s_x, s'_y)$$

satisfying the upper bound

$$|U(s_x | s')| \leq (2R + 1)^d \sup_{s, t \in \mathcal{S}} |u_l(s, t)| < \infty.$$

A similar property is valid for sufficiently rapidly decaying long-range interaction potentials, for example, under the condition

$$\sup_{s, t \in \mathcal{S}} |u_{|y|}(s, t)| \leq \frac{Const}{|y|^{d+1+\delta}}, \quad \delta > 0. \quad (7)$$

as well as for more general, but still uniformly summable many-body interactions. Here is one possible Wegner–Stollmann-type result concerning such random potentials.

Theorem 6 *Let $\Lambda \subset \mathbb{Z}^d$ be a finite subset of the lattice, $\Lambda' \subset \mathbb{Z}^d \setminus \Lambda$ any subset disjoint with Λ (Λ' may be empty), and let $s(x, \omega)$ be a Gibbs field in Λ with continuous spins $s \in \mathcal{S} = [a, b]$ generated by a two-body interaction potential $u_l(s, t)$ satisfying condition (7), with any b.c. on $\mathbb{Z}^d \setminus \Lambda$. Consider a LSO H_Λ with random potential $V(x, \omega) = s(x, \omega)$. Then for any interval $I \subset \mathbb{R}$ of length $\epsilon > 0$, we have*

$$\mathbb{P} \{ \Sigma(H_\Lambda) \cap I \neq \emptyset \mid V(y, \cdot), y \in \Lambda' \} \leq C(V) |\Lambda|^2 \epsilon, \quad C(V) < \infty.$$

In the case of unbounded spins and/or interaction potentials, the uniform boundedness of conditional single-spin distributions does not necessarily hold, since the energy of interaction of a given spin $s(0)$ with the external configuration s' may be arbitrarily large (depending on a particular form of interaction) and even *infinite*, if $s'(y) \rightarrow \infty$ too fast. In such situations, our general condition (5) may still apply, provided that rapidly growing configurations s' have sufficiently small probability, so that the outer integral in the r.h.s. of (5) converges.

7 Conclusion

Wegner–Stollmann-type estimate of the density of states in finite volumes is a key ingredient of the MSA of spectra of random Schrödinger (and some other) operators. The proposed simple extension of Stollmann’s lemma shows that a very general assumption on correlated random fields generating potential rules out an abnormal accumulation of eigen-values in finite volumes. This extension applies also to multi-particle systems [10].

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